ASYMPTOTIC BEHAVIORS OF THE COLORED JONES POLYNOMIALS OF A TORUS KNOT

HITOSHI MURAKAMI

Abstract. We study the asymptotic behaviors of the colored Jones polynomials of torus knots. Contrary to the works by R. Kashaev, O. Tirkkonen, Y. Yokota, and the author, they do not seem to give the volumes or the Chern–Simons invariants of the three-manifolds obtained by Dehn surgeries. On the other hand it is proved that in some cases the limits give the inverse of the Alexander polynomial.

Let K be a knot in the three-sphere and $J_N(K;t)$ the colored Jones polynomial corresponding to the N-dimensional representation of $sl_2(\mathbb{C})$ normalized so that $J_N(\text{unknot};t) = 1$ [8, 12]. R. Kashaev found a series of link invariants parameterized by positive integers [9] and proposed a conjecture that the asymptotic behavior of his invariants would determine the hyperbolic volume of the knot complement for any hyperbolic knot [10]. It turned out [19] that Kashaev's invariant with parameter N is equal to $\left|J_N\left(K;e^{2\pi\sqrt{-1}/N}\right)\right|$. Kashaev's conjecture was generalized to the volume conjecture which states that the asymptotic behavior of Kashaev's invariant would determine the Gromov norm [5] of the knot complement for any knot.

Kashaev and O. Tirkkonen [11] proved the volume conjecture for torus knots. More precisely they proved that for the torus knot $T(a,b)$ for coprime integers a and b, $\lim\limits_{N\to\infty} \log J_N\left(T(a,b), e^{2\pi\sqrt{-1}/N}\right)\Big/ N = 0$. Note that since the complement of a torus knot is a Seifert fibered space [17], its Gromov norm is zero.

The volume conjecture (or Kashaev's conjecture in this case) is also proved for the figure-eight knot by T. Ekholm (see for example [18]). Moreover in [20] Y. Yokota and the author proved that for the figure-eight knot E the asymptotic behavior of $J_N\left(E;e^{2\pi r\sqrt{-1}/N}\right)$ determines the volume and the Chern–Simons invariant for the closed three-manifold obtained by a Dehn surgery along E, where r is a complex parameter near 1. More precisely $\lim_{N\to\infty} \log J_N\left(E;e^{2\pi r\sqrt{-1}/N}\right)\Big/ N$ is an analytic function of r that is almost the potential function of W. Neumann and D. Zagier [21] (see also [23]).

So one could expect that the colored Jones polynomials of a knot would determine the volumes and the Chern–Simons invariants for three-manifolds obtained by Dehn surgeries along any knot. In this paper, contrary to the results above, we show that the limit of $\log J\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)\Big/ N$ behaves strangely; it cannot be extended to a continuous function of r. Note that a three-manifold obtained from a torus knot by Dehn surgery is either a Seifert fibered space, a lens space, or the connected sum of two lens spaces [17] and the Gromov norm of such a manifold is 0.

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1. Statement of the result

Let $T(a,b)$ be the (a,b)-torus knot for coprime integers a and b ($a>1$, $b>1$). Then its colored Jones polynomial $J_N\bigl(T(a,b);t\bigr)$ is given as follows [16].

$$J_N\bigl(T(a,b);t\bigr)=\frac{t^{-ab(N^2-1)/4}}{t^{N/2}-t^{-N/2}}\sum_{\varepsilon=\pm1}\sum_{k=-(N-1)/2}^{(N-1)/2}\varepsilon t^{abk^2+k(a+\varepsilon b)+\varepsilon/2}.$$

We will show the following theorem.

Theorem 1.1. *There exists a neighborhood U of $1\in\mathbb{C}$ such that if $r\in U$ and $r\notin\mathbb{R}$, then*

$$\lim_{N\to\infty}J_N\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)=\frac{1}{\Delta\left(T(a,b);e^{2\pi r\sqrt{-1}}\right)}\quad\text{if }\operatorname{Im}r<0,\tag{1.1}$$

and

$$\lim_{N\to\infty}\frac{\log J_N\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)}{N}=\left(1-\frac{1}{2abr}-\frac{abr}{2}\right)\pi\sqrt{-1}\quad\text{if }\operatorname{Im}r>0.\tag{1.2}$$

Remark 1.2. To be precise, the limit in (1.2) means

$$J_N\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)\underset{N\to\infty}{\sim}P(N)e^{N(1-1/(2abr)-abr/2)\pi\sqrt{-1}}$$

for some function of $P(N)$ such that $N^{-c}<|P(N)|<N^{c}$ for some positive integer c. Here $f(N)\underset{N\to\infty}{\sim}g(N)$ means that $\lim_{N\to\infty}f(N)/g(N)=1$. Note that $\lim_{N\to\infty}\log P(N)/N=0$ for such $P(N)$. Note also that since the real part of the right hand side of (1.2) is positive, the formula shows that $\left|J_N\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)\right|$ grows exponentially if $\operatorname{Im}r>0$.

Remark 1.3. For $r=1$, Kashaev and Tirkkonen [11] proved that

$$\left|J_N\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)\right|\underset{N\to\infty}{\sim}Ck^{3/2}$$

for some constant C. K. Hikami and A. Kirillov study the phase factor [7] and show a relation to the $SU(2)$ Chern–Simons invariant.

2. Proof

When $t=e^{2\pi r\sqrt{-1}/N}$ for $r\in\mathbb{C}\setminus\mathbb{Z}$, $J_N\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)$ can be given by the following integral form [11, Lemma 1].

$$J_N\left(T(a,b),e^{2\pi r\sqrt{-1}/N}\right)=\Phi_{a,b,r}(N)\int_C e^{Nf_{a,b,r}(z)}\tau_{a,b}(z)dz.$$

Here

$$\Phi_{a,b,r}(N):=g_{a,b,r}\sqrt{N}e^{-\left(ab\left(N^2-1\right)+a/b+b/a\right)\pi r\sqrt{-1}/(2N)}$$

$$\text{with }g_{a,b,r}:=\frac{\sqrt{ab}}{2\pi\sqrt{2r}e^{\pi\sqrt{-1}/4}\sinh\left(\pi r\sqrt{-1}\right)},$$

$$f_{a,b,r}(z):=ab\left(z-\frac{z^2}{2\pi r\sqrt{-1}}\right),$$

$$\tau_{a,b}(z) := \frac{2\sinh(az)\sinh(bz)}{\sinh(abz)},$$

and C is the line

$$C := \{se^{\varphi\sqrt{-1}} \mid s \in \mathbb{R}\} \tag{2.1}$$

with $\operatorname{Re}\left(r\sqrt{-1}e^{-2\varphi\sqrt{-1}}\right) > 0$. (We will choose φ later.)

Let $\mathcal{P}$ be the set of the poles of $\tau_{a,b}(z)$, that is , we put

$$\mathcal{P} := \left\{\frac{k\pi\sqrt{-1}}{ab}\middle| k \in \mathbb{Z}, a \nmid k, b \nmid k\right\}.$$

We will show that the result follows if $r \notin \mathbb{R}$, $\operatorname{Re} r > 0$, and $|r| > \frac{1}{ab}$.

Put $\theta := \arg r$ $(-\pi/2 < \theta < \pi/2)$ so that $r = |r|e^{\theta\sqrt{-1}}$. Put also $\varphi := \theta/2 + \pi/4$ in (2.1). Note that $\operatorname{Re}\left(r\sqrt{-1}e^{-2\varphi\sqrt{-1}}\right) = \operatorname{Re}|r| > 0$ and that $0 < \varphi < \pi/2$.

Let C' be the line parallel to C that passes through the point $\pi r\sqrt{-1}$. (We change C' slightly near $\mathcal{P}$ to avoid poles of $\tau_{a,b}(z)$ if necessary.) For a positive number R, let D_+ (D_-, respectively) be the segment of the line $\operatorname{Re} z = R$ ($\operatorname{Re} z = -R$, respectively) between C and C' oriented upward (downward, respectively). We see that C' and $D_\pm$ are parameterized as follows.

$$C' = \left\{se^{(\theta/2+\pi/4)\sqrt{-1}} + \pi r\sqrt{-1} \;\middle|\; s \in \mathbb{R}\right\},$$

$$D_\pm = \left\{ \pm R + s\sqrt{-1} \;\middle|\; \pm R\tan(\theta/2+\pi/4) \le s \le \pm R\tan(\theta/2+\pi/4) + \pi|r|h(\theta)\right\},$$

where $h(\theta) := \cos\theta + \sin\theta\tan(\theta/2+\pi/4)$ Note that C' crosses the imaginary axis at $\pi|r|h(\theta)\sqrt{-1}$.

Then by the residue theorem

$$\begin{aligned}
&\int_{\overline{C}} e^{Nf_{a,b,r}(z)}\tau_{a,b}(z)\,dz \\
&\quad= \int_{\overline{C'}} e^{Nf_{a,b,r}(z)}\tau_{a,b}(z)\,dz + \sum_k \operatorname{Res}\left(e^{Nf_{a,b,r}(z)}\tau_{a,b}(z); z = k\pi\sqrt{-1}/(ab)\right) \\
&\qquad - \int_{D_+} e^{Nf_{a,b,r}(z)}\tau_{a,b}(z)\,dz - \int_{D_-} e^{Nf_{a,b,r}(z)}\tau_{a,b}(z)\,dz,
\end{aligned}$$

where $\operatorname{Res}(F(z); z = \zeta)$ is the residue of $F(z)$ around $z = \zeta$, $\overline{C}$ ($\overline{C'}$, respectively) is the segment in C (C', respectively) bounded by $D_\pm$ oriented from left to right, and k runs over integers that are not multiples of a or b such that $k\pi\sqrt{-1}/ab$ is between C and C', that is, $0 < k < ab|r|h(\theta)$.

Since it is not hard to see that on $D_\pm$

$$\left|e^{Nf_{a,b,r}(z)}\right| < e^{abN\left(c_2R^2+c_1R+c_0\right)}$$

for some c_i with $c_2 < 0$ if R is sufficiently large, and that

$$\begin{aligned}
|\tau_{a,b}(z)| &= \frac{\left|\left(e^{az} - e^{-az}\right)\left(e^{bz} - e^{-bz}\right)\right|}{\left|e^{abz} - e^{-abz}\right|} \\
&\le \frac{\left|e^{(a+b)z}\right| + \left|e^{(a-b)z}\right| + \left|e^{(b-a)z}\right| + \left|e^{-(a+b)z}\right|}{\left|\left|e^{abz}\right| - \left|e^{-abz}\right|\right|} \\
&= \frac{e^{\pm(a+b)R} + e^{\pm(a-b)R} + e^{\pm(b-a)R} + e^{\mp(a+b)R}}{\left|e^{\pm abR} - e^{\mp abR}\right|},
\end{aligned}$$

the integrals on $D_\pm$ vanish when $R \to \infty$. Therefore we have

$$\int_C e^{Nf_{a,b,r}(z)}\tau_{a,b}(z)\,dz$$

$$= \int_{C'} e^{N f_{a,b,r}(z)} \tau_{a,b}(z)\, dz + \sum_k \operatorname{Res}\left(e^{N f_{a,b,r}(z)} \tau_{a,b}(z); z = k\pi\sqrt{-1}/(ab)\right).$$

We will apply the saddle point method (or the method of steepest descent) to know the asymptotic behavior of the integral on C'. See for example [14, Theorem 7.2.9]. On C' we have

$$f_{a,b,r}\left(se^{(\theta/2+\pi/4)\sqrt{-1}} + \pi r\sqrt{-1}\right) = -\frac{s^2}{2\pi|r|} + \frac{\pi r\sqrt{-1}}{2}.$$

So $\operatorname{Im} f_{a,b,r}(z)$ is constant on C' and $\operatorname{Re} f_{a,b,r}(z)$ takes its unique maximum at $z = \pi r\sqrt{-1}$ (when $s = 0$) on C'. (Note that this is also true if we change C' slightly near a pole of $\tau_{a,b}$.) Therefore from the saddle point method we have

$$\begin{aligned}
&\int_{C'} e^{N f_{a,b,r}(z)} \tau_{a,b}(z)\, dz \\
&\underset{N\to\infty}{\sim} \frac{\sqrt{2\pi}\tau_{a,b}(\pi r\sqrt{-1})}{\sqrt{N}\sqrt{-d^2 f_{a,b,r}(z)/dz^2\big|_{z=\pi r\sqrt{-1}}}} e^{N f_{a,b,r}(\pi r\sqrt{-1})} \\
&= \pi\sqrt{\frac{2r}{abN}} e^{\pi\sqrt{-1}/4} \tau_{a,b}(\pi r\sqrt{-1}) e^{N ab\pi r\sqrt{-1}/2}.
\end{aligned}$$

We also have

$$\begin{aligned}
&\operatorname{Res}\left(e^{N f_{a,b,r}(z)} \tau_{a,b}(z); z = k\pi\sqrt{-1}/(ab)\right) \\
&= e^{Nk\pi\sqrt{-1}\left(1-\frac{k}{2abr}\right)} \operatorname{Res}\left(\tau_{a,b}(z); z = k\pi\sqrt{-1}/(ab)\right) \\
&= (-1)^k \frac{2\sinh(k\pi\sqrt{-1}/a)\sinh(k\pi\sqrt{-1}/b)}{ab} e^{Nk\pi\sqrt{-1}\left(1-\frac{k}{2abr}\right)}.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\int_{C} e^{N f_{a,b,r}(z)} \tau_{a,b}(z)\, dz \\
&\underset{N\to\infty}{\sim} \pi\sqrt{\frac{2r}{abN}} e^{\pi\sqrt{-1}/4} \tau_{a,b}(\pi r\sqrt{-1}) e^{N ab\pi r\sqrt{-1}/2} \\
&\quad + \sum_k (-1)^k \frac{2\sinh(k\pi\sqrt{-1}/a)\sinh(k\pi\sqrt{-1}/b)}{ab} e^{Nk\pi\sqrt{-1}\left(1-\frac{k}{2abr}\right)}.
\end{aligned} \tag{2.2}$$

Now we want to know the biggest real part of the exponents in (2.2). We have

$$\begin{aligned}
&\max\left\{\operatorname{Re}\left(\frac{ab\pi r\sqrt{-1}}{2}\right), \max_k \operatorname{Re}\left(k\pi\sqrt{-1}\left(1-\frac{k}{2abr}\right)\right)\right\} \\
&= \max\left\{-\frac{ab\pi \operatorname{Im} r}{2}, \max_k \frac{-k^2\pi \operatorname{Im} r}{2ab|r|^2}\right\} \\
&= \begin{cases} \dfrac{\pi \operatorname{Im} r}{2ab} \max\limits_k \left\{-a^2b^2, -\dfrac{k^2}{|r|^2}\right\} & \text{if } \operatorname{Im} r > 0 \\[2ex] -\dfrac{\pi \operatorname{Im} r}{2ab} \max\limits_k \left\{a^2b^2, \dfrac{k^2}{|r|^2}\right\} & \text{if } \operatorname{Im} r < 0 \end{cases}.
\end{aligned}$$

Since $1 < h(\theta)$ when $0 < \theta < \pi/2$, $|r| > 1/(ab)$, and k runs over positive integers less than $ab|r|h(\theta) > 1$, we have

$$\max_k \left\{-a^2b^2, -\frac{k^2}{|r|^2}\right\} = -\frac{1}{|r|^2}$$

if $\operatorname{Im} r > 0$. Since $0 < h(\theta) < 1$ when $-\pi/2 < \theta < 0$, $0 < k < ab|r|h(\theta) < ab|r|$. So we have

$$\max_k \left\{ a^2 b^2, \frac{k^2}{|r|^2} \right\} = a^2 b^2$$

if $\operatorname{Im} r < 0$.

Therefore if $\operatorname{Im} r < 0$

$$\int_C e^{N f_{a,b,r(z)}} \tau_{a,b}(z) dz \underset{N\to\infty}{\sim} \pi \sqrt{\frac{2r}{abN}} e^{\pi\sqrt{-1}/4} \tau_{a,b}(\pi r \sqrt{-1}) e^{N ab \pi r \sqrt{-1}/2}$$

and if $\operatorname{Im} r > 0$

$$\int_C e^{N f_{a,b,r}(z)} \tau_{a,b}(z)\, dz \underset{N\to\infty}{\sim} -\frac{2\sinh(\pi\sqrt{-1}/a)\sinh(\pi\sqrt{-1}/b)}{ab} e^{N\pi\sqrt{-1}\left(1-\frac{1}{2abr}\right)}.$$

Since

$$\Phi_{a,b,r}(N) \underset{N\to\infty}{\sim} g_{a,b,r}\sqrt{N} e^{-N abr\pi\sqrt{-1}/2},$$

we have

$$\begin{aligned} &J_N\big(T(a,b); e^{2\pi r\sqrt{-1}}\big) \\ &\quad\underset{N\to\infty}{\sim} -\frac{2\sinh(\pi\sqrt{-1}/a)\sinh(\pi\sqrt{-1}/b) g_{a,b,r}}{ab} \sqrt{N} e^{N\pi\sqrt{-1}\left(1-abr/2-1/(2abr)\right)} \end{aligned}$$

if $\operatorname{Im} r > 0$ and

$$J_N\big(T(a,b); e^{2\pi r\sqrt{-1}}\big) \underset{N\to\infty}{\sim} \frac{\sinh(ar\pi\sqrt{-1})\sinh(br\pi\sqrt{-1})}{\sinh(abr\pi\sqrt{-1})\sinh(r\pi\sqrt{-1})}$$

if $\operatorname{Im} r < 0$.

Since the Alexander polynomial $\Delta\big(T(a,b);t\big)$ of $T(a,b)$ is $\frac{(t^{ab/2}-t^{-ab/2})(t^{1/2}-t^{-1/2})}{(t^{a/2}-t^{-a/2})(t^{b/2}-t^{-b/2})}$ (see for example [13, Page 119]), we have

$$\lim_{N\to\infty} J_N\left(T(a,b); e^{2\pi r\sqrt{-1}/N}\right) = \frac{1}{\Delta\left(T(a,b); e^{2\pi r\sqrt{-1}}\right)}$$

if $\operatorname{Im} r < 0$. If $\operatorname{Im} r > 0$ we have

$$\lim_{N\to\infty} \frac{\log J_N\left(T(a,b); e^{2\pi r\sqrt{-1}/N}\right)}{N} = \left(1 - \frac{1}{2abr} - \frac{abr}{2}\right)\pi\sqrt{-1}.$$

This completes the proof of Theorem 1.1.

3. Comments

Here are a couple of probably inaccurate comments about the relation of the limit of the colored Jones polynomials of a torus knot and its Alexander polynomial.

Comment 3.1. The Melvin–Morton–Rozansky conjecture proved by D. Bar-Natan and S. Garoufalidis [1], which was proposed by P. Melvin and H. Morton [15] and 'proved' non-rigorously by L. Rozansky [22], tells that if we expand the colored Jones polynomial of a knot as a power series

$$J_N\left(K; e^h\right) = \sum_{d,l\geq 0} b_{dl}(K) h^d N^l,$$

then

(i) $b_{dl}(K) = 0$ if $l > d$,

(ii) the diagonal coefficients give the inverse of the Alexander polynomial, that is,

$$\sum_{d\ge 0} b_{dd}(K)(hN)^d = \frac{1}{\Delta(K;e^{hN})},$$

where $\Delta(K;t)$ is the Alexander polynomial of K.

If we replace h with $2\pi r\sqrt{-1}/N$, we have from (i)

$$J_N\left(K;e^{2\pi r\sqrt{-1}/N}\right) = \sum_{d\ge l\ge 0} b_{dl}(K)(2\pi r\sqrt{-1})^d N^{l-d} \tag{3.1}$$

and from (ii)

$$\sum_{d\ge 0} b_{dd}(K)(2\pi r\sqrt{-1})^d = \frac{1}{\Delta(K;e^{2\pi r\sqrt{-1}})}. \tag{3.2}$$

Theorem 1.1 would say that if $\operatorname{Im} r < 0$ and K is a torus knot, then the right hand side of (3.1) converges to the left hand side of (3.2) and gives the inverse of the Alexander polynomial.

Note that this does not hold for the figure-eight knot [20].

Comment 3.2. In [6, Theorem 4] Hikami obtains a recursive formula for the colored Jones polynomials of a torus knot:

$$\begin{aligned} &J_N\big(T(a,b);t\big) \\ &= \frac{t^{(a-1)(b-1)(1-N)/2}}{1-t^{-N}}\left(1-t^{a(1-N)-1}-t^{b(1-N)-1}+t^{(a+b)(1-N)}\right) \\ &\qquad + \frac{1-t^{2-N}}{1-t^{-N}}t^{ab(1-N)-1}J_{N-2}\big(T(a,b),t\big). \end{aligned} \tag{3.3}$$

The limit (1.1) could be obtained from the following 'fake' calculation. If we replace t with $e^{2\pi r\sqrt{-1}/N}$ in (3.3), we could approximate

$$t^{(a-1)(b-1)(1-N)/2} = e^{r(a-1)(b-1)\pi\sqrt{-1}/N - r(a-1)(b-1)\pi\sqrt{-1}}$$

as $e^{-r(a-1)(b-1)\pi\sqrt{-1}}$ for large N. Approximating the other terms similarly, we would have

$$\begin{aligned} J_\infty = &\frac{e^{-r(a-1)(b-1)\pi\sqrt{-1}}}{1-e^{-2r\pi\sqrt{-1}}}\left(1-e^{-2ar\pi\sqrt{-1}}-e^{-2br\pi\sqrt{-1}}+e^{-2(a+b)r\pi\sqrt{-1}}\right) \\ &+ e^{-2abr\pi\sqrt{-1}}J_\infty, \end{aligned}$$

where J_∞ denotes the 'limit' of $J_N\left(T(a,b);e^{2\pi r\sqrt{-1}/N}\right)$. Then (1.1) follows easily.

Note that there exists a recursive formula for the colored Jones polynomials of any knot [2]. See also [3, 4] for a recursive formula of the $(2,2n+1)$ torus knot.

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Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152-8551, Japan

E-mail address: `starshea@tky3.3web.ne.jp`